



TITLE:

Confluence from Siegel-Whittaker functions to Whittaker functions on  $\mathrm{Sp}(2, \mathbb{R})$  (Automorphic forms on  $\mathrm{Sp}(2, \mathbb{R})$  and  $\mathrm{SU}(2, 2)$  III)

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# Confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2, \mathbf{R})$

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## 概要

この小論では、それぞれの球関数の明示公式を利用して、 $Sp(2, \mathbf{R})$  上の Siegel-Whittaker 関数から Whittaker functions への合流を論ずる。証明には、良い Whittaker (resp. Siegel-Whittaker) functions の、第 2 種の Whittaker (resp. Siegel-Whittaker) functions による展開定理を用いる。

## Abstract

We discuss a confluence from Siegel-Whittaker functions to Whittaker functions on  $Sp(2, \mathbf{R})$  by using their explicit formulae. In our proof, we use expansion theorems of the good Whittaker functions by the secondary Whittaker functions.

## 序

記号  $\pi$  によって、ある半単純実 Lie 群  $G$  の既約認容表現を表わす。  $G$  の閉部分群  $R$  とその既約ユニタリ表現  $\eta$  の対  $(R, \eta)$  が与えられたとき、平滑に誘導された  $G$ -表現  $\text{Ind}_R^G(\eta)$  に値を持つ絡空間  $\text{Hom}_G(\pi, \text{Ind}_R^G(\eta))$  あるいは  $\text{Hom}_{(\mathfrak{g}_C, K)}(\pi, \text{Ind}_R^G(\eta))$  を考える。ここで  $K$  は  $G$  の極大コンパクト部分群で  $\mathfrak{g}_C$  は  $G$  Lie 環の複素化を表わす。この空間の非零元  $I$ 、即ち表現  $\pi$  から  $\text{Ind}_R^G(\eta)$  への絡作用素 (intertwining operator)  $I$  を一つ選ぶとき、その像  $\text{Im}(I)$  は  $\pi$  の  $\text{Ind}_R^G(\eta)$  の中での一般化された球模型 (generalized spherical model) あるいは球実現 (generalized spherical realization) である。

群  $G$  が両側剰余類分解  $G = RA_RK$  を有つような  $G$  の最大  $\mathbf{R}$ -分裂カルタン部分群の分裂成分である閉部分群  $A_R$  があると、それを一つ固定する。そのとき  $\pi$  の適切な  $K$ -type  $\tau$  (通常は重複度 1 のもの) を選び、また絡空間  $\text{Hom}_{(\mathfrak{g}_C, K)}(\pi, \text{Ind}_R^G(\eta))$  は有限次元と仮定する。そのとき上の実現の  $\tau$ -等質成分  $\text{Im}(I)[\tau]$  of  $\text{Im}(I)$  の  $U(\mathfrak{g}_C)$  内の零化元 (annihilator) の  $A_R$ -動径成分からなるホロノミック系を計算する。

例えば  $G = Sp(2, \mathbf{R})$  のとき、保型形式研究に用いるという動機から、我々はいろいろな種類の表現  $\pi$  と対  $(R, \eta)$  に対して、これらのホロノミック系とその明示的な解 (即ち  $G$  上の一般化された球関数) を求めてきた (cf. [12], [13], [14], [16], [17], [22], [23], [26])。

一度同じ  $\pi$  に対して別の相異なる二つの対  $(R_1, \eta_1)$  and  $(R_0, \eta_0)$  に対して、二つの異なる実現を得たとき、当然次はこれらの二つの模型の関係を知りたくなる。例えば、直感的な説明をすると、 $G$  の球部分群  $R_t$  とその表現  $\eta_t$  の対の変形族  $(R_t, \eta_t)$  で、 $t \neq 0$  のときには各対  $(R_t, \eta_t)$  は  $(R_1, \eta_1)$  は内部自己同型による捻りで、極限  $t \rightarrow 0$  においては「縮約」対  $(R_0, \eta_0)$  になるものを考える。これは物理学者が考え始め、例えば Dooley [3] などによってきちんと取り扱われた Lie 群とその表現の contraction あるいは deformation という手法の変奏版である。

この論説では長いルートに付随する  $G$  のヤコビ極大部分群  $P_J$  による parabolic induction で得られる一般化主系列表現の二つの模型間の合流を論ずる。我々の設定では、 $(R_0, \eta_0)$  は極

大ユニポテント部分群  $R_0$  と  $R_0$  の非退化指標  $\eta_0$  の対で、それは付随する球模型が通常の Whittaker 模型  $\pi$  (cf. [18], [30], [19]) であることを意味する。そして対  $(R_1, \eta_1)$  は  $\pi$  の Siegel-Whittaker 模型を導く。これは宮崎琢也 [21] や石井卓 [16] によって調べられた。Lie 群とその表現の縮約や変形族というアイデアを厳密に展開するのは、それによってここで得られる結果に比べると重い準備が必要となる。この高いコストを回避するため、ここでは単に関連する対の変形族  $(R_t, \eta_t)$  の球関数の  $A_R$ -動径部分のみを直接に計算する。元の直感的なアイデアは第5節で、謂わば「後知恵」として記述される。

主結果は第4節にある。それぞれの球関数のホロノミック系の合流 (Theorem 4.1), 「第2種」 (secondary) 球関数の合流 (Theorem 4.2), そして最後に良い増大条件を有つ一意に定まる球関数の合流 (Theorem 4.3) という順で証明される。ここで言う secondary spherical functions とは、行列係数の漸近挙動を調べるために Harish-Chandra [7] によって考えられた、球関数のホロノミック系の無限遠における確定特異点でのべき級数解のことである (その後、Heckman-Opdam [11], [28] によってこの手法は一般化された)。さらに、この種の関数は、Miatello-Wallach [20] や Oda-Tsuzuki [27] の論文に見るように、Poincaré 級数の構成で基本的な役割を果たす。「良い球関数」の合流の証明で基本的なことは、「第2種」球関数合流の結果と、良い球関数の「第2種」球関数による展開定理をもちいることである (Theorems 2.4 and 3.4)。我々の結果は、それ自身興味深い点があると思うが (表現の変形は少なくとも最近はほとんど考えられていない)、さらに保型形式研究に、例えば無限素点での局所ゼータ関数の研究に、役に立つことを期待している (cf. [25])。

関連する周辺の問題はいろいろあるが、また別の機会に。

**注意** 用語 secondary spherical function の secondary であるが、「第2種」 (of the second kind) の意味もあるが、元はボクシングの「セコンド」のような意味で使っている。Latin 語の語源は、secundus は元は、to follow の語から来ていると言う。当面は、素直に「補助的な」の意味で受け取って下さい。

**記号** For  $a \in \mathbf{C}$  and  $n \in \mathbf{Z}$ ,  $(a)_n = \Gamma(a+n)/\Gamma(a)$  the Pochhammer symbol. For complex numbers  $a_i$  ( $1 \leq i \leq r$ ) and  $b_j$  ( $1 \leq j \leq s$ ), set

$$\Gamma[a_1, \dots, a_r] = \prod_{i=1}^r \Gamma(a_i), \quad \Gamma \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \right] = \prod_{i=1}^r \Gamma(a_i) / \prod_{i=1}^s \Gamma(b_i).$$

## 1 Preliminaries

### 1.1 Basic notions

Let  $G$  be the real symplectic group of degree two:

$$G = Sp(2, \mathbf{R}) = \left\{ g \in SL(4, \mathbf{R}) \mid {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\},$$

with  $1_2$  the unit matrix of degree two.

Fix a maximal compact subgroup  $K$  of  $G$  by

$$K = \left\{ k(A, B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \mid A, B \in M(2, \mathbf{R}) \right\}.$$

It is isomorphic to the unitary group  $U(2)$  via the homomorphism

$$K \ni k(A, B) \mapsto A + \sqrt{-1}B \in U(2).$$

Then the set of irreducible representations of  $K$  is parameterized by  $\{(\lambda_1, \lambda_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid \lambda_1 \geq \lambda_2\}$  and we denote by  $\tau_{(\lambda_1, \lambda_2)} = \text{Sym}^{\lambda_1 - \lambda_2} \otimes \det^{\lambda_2}$  the representation corresponding to  $(\lambda_1, \lambda_2)$ .

We define two spherical subgroups  $R_i$  of  $G$  and their representations. The first one is a maximal unipotent radical of  $G$  given by

$$R_1 = \left\{ n(n_0, n_1, n_2, n_3) = \left( \begin{array}{c|cc} 1 & n_0 & \\ \hline & 1 & \\ & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left( \begin{array}{c|cc} 1 & n_1 & n_2 \\ \hline & 1 & n_2 & n_3 \\ & & 1 & \\ & & & 1 \end{array} \right) \mid n_i \in \mathbf{R} \right\}.$$

Any unitary character  $\eta_1$  of  $R_1$  can be written as

$$\eta_1(n(n_0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3))$$

with some  $c_0, c_3 \in \mathbf{R}$ . In this paper we assume  $\eta_1$  is non-degenerate, that is,  $c_0 c_3 \neq 0$ . Taking a maximal split torus  $A$  of  $G$  by

$$A = \{a(a_1, a_2) = \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0\},$$

we have the Iwasawa decomposition  $G = R_1 A K$ .

The second spherical subgroup  $R_2$  is defined as follows. Let  $P_S = L_S \ltimes N_S$  be the Siegel parabolic subgroup with the Levi part  $L_S$  and the abelian unipotent radical  $N_S$  given by

$$L_S = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \mid A \in GL(2, \mathbf{R}) \right\},$$

$$N_S = \{n(0, n_1, n_2, n_3) \mid n_1, n_2, n_3 \in \mathbf{R}\}.$$

Fix a non-degenerate unitary character  $\xi$  of  $N_S$  by

$$\xi(n(0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1} \text{Tr}(H_\xi T))$$

with  $T = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix}$ ,  $H_\xi = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix} \in M(2, \mathbf{R})$  and  $\det H_\xi \neq 0$ . Consider the action of  $L_S$  on  $N_S$  by conjugation and the induced action on the character group  $\widehat{N_S}$ . Define  $SO(\xi)$  to be the identity component of the subgroup of  $L_S$  which stabilize  $\xi$ :

$$SO(\xi) := \text{Stab}_{L_S}(\xi)^\circ = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \mid {}^t A H_\xi A = H_\xi \right\}.$$

Then  $SO(\xi)$  is isomorphic to  $SO(2)$  if  $\det H_\xi > 0$  and to  $SO_o(1, 1)$  if  $\det H_\xi < 0$ . In this paper we treat the case that  $\xi$  is a 'definite' character, that is,  $\det H_\xi > 0$ . So we may assume  $h_1, h_2 > 0$  and  $h_3 = 0$  without loss of generality. We sometimes identify the element of  $SO(\xi)$  with its upper left  $2 \times 2$  component. Fix a unitary character  $\chi_{m_0}$  ( $m_0 \in \mathbf{Z}$ ) of  $SO(\xi) \cong SO(2)$  by

$$\chi_{m_0} \left( \begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix}^{-1} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix} \right) = \exp(\sqrt{-1} m_0 \theta).$$

We define  $R_2 = SO(\xi) \ltimes N_S$  and  $\eta_2 = \chi_{m_0} \boxtimes \xi$ . Note that we also have the decomposition  $G = R_2 A K$ .

## 1.2 Spherical functions

For the pair  $(R_i, \eta_i)$  defined as above, consider the space  $C_{\eta_i}^\infty(R_i \backslash G)$  of complex valued  $C^\infty$  functions  $f$  on  $G$  satisfying

$$f(rg) = \eta_i(r)f(g) \quad \text{for all } (r, g) \in R_i \times G.$$

By the right translation,  $C_{\eta_i}^\infty(R_i \backslash G)$  is a smooth  $G$ -module and we denote by the same symbol its underlying  $(\mathfrak{g}_\mathbb{C}, K)$ -module ( $\mathfrak{g}_\mathbb{C}$  is the complexification of the Lie algebra of  $G$ ). For an irreducible admissible representation  $(\pi, H_\pi)$  of  $G$  and the subspace  $H_{\pi, K}$  of  $K$ -finite vectors, the intertwining space

$$\mathcal{I}_{\eta_i, \pi} = \text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(H_{\pi, K}, C_{\eta_i}^\infty(R_i \backslash G))$$

between the  $(\mathfrak{g}_\mathbb{C}, K)$ -modules is called the space of *algebraic Whittaker functionals* for  $i = 1$ , or *algebraic Siegel-Whittaker functionals* for  $i = 2$ . For a finite-dimensional  $K$ -module  $(\tau, V_\tau)$ , denote by  $C_{\eta_i, \tau}^\infty(R_i \backslash G/K)$  the space

$$\{\phi : G \rightarrow V_\tau, C^\infty \mid \phi(r g k) = \eta_i(r)\tau(k^{-1})\phi(g), \text{ for all } (r, g, k) \in R_i \times G \times K\}.$$

Let  $(\tau^*, V_{\tau^*})$  be a  $K$ -type of  $\pi$  and  $\iota : V_{\tau^*} \rightarrow H_\pi$  be an injection. Here  $\tau^*$  means the contragredient representation of  $\tau$ . Then for  $\Phi \in \mathcal{I}_{\eta_i, \pi}$ , we can find an element  $\phi_i$  in

$$C_{\eta_i, \tau}^\infty(R_i \backslash G/K) = C_{\eta_i}^\infty(R_i \backslash G) \otimes V_{\tau^*} \cong \text{Hom}_K(V_{\tau^*}, C_{\eta_i}^\infty(R_i \backslash G))$$

via  $\Phi(\iota(v^*))(g) = \langle v^*, \phi_i(g) \rangle$  with  $\langle \cdot, \cdot \rangle$  the canonical pairing on  $V_{\tau^*} \times V_\tau$ .

Since there is the decomposition  $G = R_i A K$ , our (generalized) spherical function  $\phi_i$  is determined by its restriction  $\phi_i|_A$  to  $A$ , which we call the *radial part* of  $\phi_i$ . For a subspace  $X$  of  $C_{\eta_i, \tau}^\infty(R_i \backslash G/K)$ , we denote  $X|_A = \{\phi|_A \in C^\infty(A) \mid \phi \in X\}$ .

Let us define two spaces  $\text{Wh}(\pi, \eta_1, \tau)$  and  $\text{SW}(\pi, \eta_2, \tau)$  of spherical functions and their subspaces  $\text{Wh}(\pi, \eta_1, \tau)^{\text{mod}}$  and  $\text{SW}(\pi, \eta_2, \tau)^{\text{rap}}$  as follows:

$$\text{Wh}(\pi, \eta_1, \tau) = \bigcup_{\iota \in \text{Hom}_K(\tau^*, \pi)} \{\phi_i \mid \Phi \in \mathcal{I}_{\eta_1, \pi}\},$$

$$\text{Wh}(\pi, \eta_1, \tau)^{\text{mod}} = \{\phi_i \in \text{Wh}(\pi, \eta_1, \tau) \mid \phi_i|_A \text{ is of moderate growth as } a_1, a_2 \rightarrow \infty\},$$

$$\text{SW}(\pi, \eta_2, \tau) = \bigcup_{\iota \in \text{Hom}_K(\tau^*, \pi)} \{\phi_i \mid \Phi \in \mathcal{I}_{\eta_2, \pi}\},$$

and

$$\text{SW}(\pi, \eta_2, \tau)^{\text{rap}} = \{\phi_i \in \text{SW}(\pi, \eta_2, \tau) \mid \phi_i|_A \text{ decays rapidly as } a_1, a_2 \rightarrow \infty\}.$$

We call an element in  $\text{Wh}(\pi, \eta_1, \tau)$  (resp.  $\text{SW}(\pi, \eta_2, \tau)$ ) a *Whittaker function* (resp. *Siegel-Whittaker function*) for  $(\pi, \eta_i, \tau)$ .

As we shall see in the next two sections, radial parts of spherical functions satisfy certain holonomic systems of regular singular type. We call the power series solutions at the regular singularities of the systems *secondary spherical functions*, and the elements of  $\text{Wh}(\pi, \eta_1, \tau)^{\text{mod}}$  and  $\text{SW}(\pi, \eta_2, \tau)^{\text{rap}}$  *good spherical functions*.

### 1.3 $P_J$ -principal series representations

In this section we recall the generalized principal series representations of  $G$  associated with the Jacobi maximal parabolic subgroup  $P_J$  of  $G$  corresponding to the long root. A Langlands decomposition  $P_J = M_J A_J N_J$  is given by

$$M_J = \left\{ \left( \begin{array}{c|c} \epsilon & \\ \hline a & b \\ \hline c & d \end{array} \right) \mid \epsilon \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}) \right\},$$

$$A_J = \{a(a_1, 1) = \text{diag}(a_1, 1, a_1^{-1}, 1) \in A \mid a_1 > 0\},$$

$$N_J = \{n(n_0, n_1, n_2, 0) \in N = R_1 \mid n_i \in \mathbf{R}\}.$$

A discrete series representation  $(\sigma, V_\sigma)$  of the semisimple part  $M_J \cong \{\pm 1\} \times SL(2, \mathbf{R})$  of  $P_J$  is of the form  $\sigma = \varepsilon \boxtimes D_k^\pm$  ( $k \geq 2$ ), where  $\varepsilon : \{\pm 1\} \rightarrow \mathbf{C}^*$  is a character and  $D_k^+$  (resp.  $D_k^-$ ) is the discrete series representation of  $SL(2, \mathbf{R})$  with Blattner parameter  $k$  (resp.  $-k$ ). For  $\nu \in \mathbf{C}$ , define a quasi-character  $\exp(\nu)$  of  $A_J$  by  $\exp(\nu)(a(a_1, 1)) = a_1^\nu$ . We call an induced representation

$$I(P_J; \sigma, \nu) = C^\infty\text{-Ind}_{P_J}^G(\sigma \otimes \exp(\nu + 1) \otimes 1_{N_J})$$

the  $P_J$ -principal series representation of  $G$ .

The  $K$ -types of  $I(P_J; \sigma, \nu)$  is fully described in [23, Proposition 2.1] and [13, Proposition 2.3]. In particular, if  $\pi = I(P_J; \varepsilon \boxtimes D_k^+, \nu)$  with  $\varepsilon(\text{diag}(-1, 1, -1, 1)) = (-1)^k$  (even  $P_J$ -principal series), then the corner  $K$ -type  $\tau^* = \tau_{(k,k)}$  occurs in  $\pi$  with multiplicity one.

## 2 Whittaker functions

### 2.1 Basic results

Let  $\pi = I(P_J; \varepsilon \boxtimes D_k^+, \nu)$  be an irreducible even  $P_J$ -principal series representation of  $G$  with  $\varepsilon(\text{diag}(-1, 1, -1, 1)) = (-1)^k$ , and  $\tau^* = \tau_{(k,k)}$  is the corner  $K$ -type of  $\pi$ . We first prepare some basic facts on the Whittaker functions for  $(\pi, \eta_1, \tau)$ . Throughout this section we use a coordinate  $x = (x_1, x_2)$  on  $A$  defined by

$$x_1 = \left( \pi c_0 \frac{a_1}{a_2} \right)^2, \quad x_2 = 4\pi c_3 a_2^2.$$

By combining the results of Kostant ([18, §6]), Wallach ([30, Theorem 8.8]), Matumoto ([19, Theorem 6.2.1]) and Miyazaki and Oda ([23, Proposition 7.1, Theorem 8.1]), we obtain

**Proposition 2.1.** *Let  $\pi$  and  $\tau$  be as above. Then we have the following:*

(i) *We have  $\dim \mathcal{I}_{\eta_1, \pi} = \dim \text{Wh}(\pi, \eta_1, \tau) = 4$ , and a function*

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) h_W(a)$$

*on  $A$  is in the space  $\text{Wh}(\pi, \eta_1, \tau)|_A$  if and only if  $h_W(a) = h_W(x)$  is a smooth solution of the following holonomic system of rank 4:*

$$(2.1) \quad \left\{ \partial_{x_1} \left( -\partial_{x_1} + \partial_{x_2} + \frac{1}{2} \right) + x_1 \right\} h_W(x) = 0,$$

$$(2.2) \quad \left\{ \left( \partial_{x_2} + \frac{k+\nu}{2} \right) \left( \partial_{x_2} + \frac{k-\nu}{2} \right) - x_2 \left( -\partial_{x_1} + \partial_{x_2} + \frac{1}{2} \right) \right\} h_W(x) = 0,$$

where  $\partial_{x_i} = x_i(\partial/\partial x_i)$  ( $i = 1, 2$ ) is the Euler operator with respect to  $x_i$ .

(ii)  $\dim \text{Wh}(\pi, \eta_1, \tau)^{\text{mod}} \leq 1$ . Moreover this inequality is an equality if and only if  $c_3 > 0$ .

**Remark 1.** Since [23] treated the case  $\sigma = \varepsilon \boxtimes D_k^-$ , we need a minor change by using the explicit formulas of ‘shift operators’ ([22, Proposition 8.3]).

## 2.2 Explicit formulas of secondary Whittaker functions

In this section we determine the space of smooth solutions of the holonomic system in Proposition 2.1, therefore the space of the Whittaker functions  $\text{Wh}(\pi, \eta_1, \tau)$ , explicitly. Set

$$h_W(a) = h_W(x) = \sum_{m,n \geq 0} a_{m,n} x_1^{\sigma_1+m} x_2^{\sigma_2+n}$$

with  $a_{0,0} \neq 0$ . Then we have the following difference equations for  $\{a_{m,n}\}$ :

$$(2.3) \quad (\sigma_1 + m) \left\{ -(\sigma_1 + m) + (\sigma_2 + n) + \frac{1}{2} \right\} a_{m,n} + a_{m-1,n} = 0,$$

$$(2.4) \quad \left( \sigma_2 + n + \frac{k+\nu}{2} \right) \left( \sigma_2 + n + \frac{k-\nu}{2} \right) a_{m,n} + \left\{ (\sigma_1 + m) - (\sigma_2 + n) + \frac{1}{2} \right\} a_{m,n-1} = 0.$$

Here we promise  $a_{m,n} = 0$  if  $m < 0$  or  $n < 0$ . By putting  $m = n = 0$  in (2.3) and (2.4), we can find the characteristic indices

$$(\sigma_1, \sigma_2) = \left( 0, \frac{-k \pm \nu}{2} \right), \left( \frac{-k \pm \nu + 1}{2}, \frac{-k \pm \nu}{2} \right).$$

If  $\nu$  is not an integer, we can determine the coefficients  $a_{m,n}$  inductively for each case and thus obtain

**Proposition 2.2.** (cf. [29, Proposition 2.1]) For  $\nu \notin \mathbf{Z}$ , define the functions  $h_W^i(\nu; a) = h_W^i(\nu; x)$  on  $A$  by

$$\begin{aligned} h_W^1(\nu; x) &= \sum_{m,n \geq 0} c_{m,n}^1 x_1^m x_2^{n+(-k+\nu)/2}, \\ h_W^2(\nu; x) &= \sum_{m,n \geq 0} c_{m,n}^2 x_1^{m+n+(-k+\nu+1)/2} x_2^{n+(-k+\nu)/2}, \end{aligned}$$

with

$$\begin{aligned} c_{m,n}^1 &= \Gamma \left[ \begin{matrix} -n-\nu, & -m+n+\frac{-k+\nu+1}{2} \\ -\nu, & \frac{-k+\nu+1}{2} \end{matrix} \right] \frac{(-1)^{m+n}}{m! n!}, \\ c_{m,n}^2 &= \Gamma \left[ \begin{matrix} -n-\nu, & -m-n+\frac{k-\nu-1}{2} \\ -\nu, & \frac{k-\nu-1}{2} \end{matrix} \right] \frac{(-1)^{m+n}}{m! n!}. \end{aligned}$$

Then the power series  $h_W^i(\nu; x)$  converges for any  $x \in \mathbf{C}^2$  and the set  $\{h_W^i(\varepsilon\nu; x) \mid i = 1, 2, \varepsilon \in \{\pm 1\}\}$  forms a basis of the space of solutions of the system in Proposition 2.1.

### 2.3 Explicit formulas of good Whittaker functions

When  $c_3 < 0$ , Proposition 2.1 tells us that there is no non-zero moderate growth Whittaker function. Therefore let us assume  $c_3 > 0$  in the following discussion. The integral expression for the Whittaker functions of moderate growth was obtained by Miyazaki and Oda.

**Proposition 2.3.** ([23, Theorem 8.1]) *Let  $\pi$  and  $\tau$  be as before. Define*

$$g_W(a) = g_W(x) := x_2^{-1/2} \int_0^\infty t^{-k+1/2} W_{0,\nu}(t) \exp\left(-\frac{t^2}{16x_2} - \frac{16x_1x_2}{t^2}\right) \frac{dt}{t},$$

with  $W_{\kappa,\mu}$  the classical Whittaker function. Then the function

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) g_W(a)$$

gives a non-zero element in  $\text{Wh}(\pi, \eta_1, \tau)^{\text{mod}}|_A$  which is unique up to constant multiple.

### 2.4 Expansion theorem for Whittaker functions

Now we express the moderate growth Whittaker function  $g_W$  as a linear combination of  $h_W^i$ .

**Theorem 2.4.** *For  $\nu \notin \mathbf{Z}$ , let  $h_W^i(\nu; a)$  and  $g_W(a)$  be the function defined in Proposition 2.2 and 2.3, respectively. Then*

$$g_W(a) = c_W \sum_{\varepsilon \in \{\pm 1\}} \left( \Gamma\left[-\varepsilon\nu, \frac{-k + \varepsilon\nu + 1}{2}\right] h_W^1(\varepsilon\nu; a) + \Gamma\left[-\varepsilon\nu, \frac{k - \varepsilon\nu - 1}{2}\right] h_W^2(\varepsilon\nu; a) \right)$$

with  $c_W = 2^{1-2k} \pi^{-1/2}$ .

## 3 Siegel-Whittaker functions

### 3.1 Basic results

Miyazaki ([21]) studied the Siegel-Whittaker functions for  $P_J$ -principal series and obtained the multiplicity one property and the explicit integral representation for rapidly decreasing function. As in the previous section, we introduce the coordinate  $y = (y_1, y_2)$  on  $A$  by

$$y_1 = \frac{h_1 a_1^2}{h_2 a_2^2}, \quad y_2 = 4\pi h_2 a_2^2.$$

We remark on a compatibility condition. For a non-zero element  $\phi$  of  $C_{\eta_i, \tau_{(-k, -k)}}^\infty(R_i \backslash G/K)$ , we have

$$\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \boxtimes \xi)(m) \tau_{(-k, -k)}(m) \phi(a),$$

where  $a \in A$  and  $m \in SO(\xi) \cap Z_K(A) = \{\pm 1_4\}$ . If we take  $m = -1_4$ ,  $(\chi_{m_0} \boxtimes \xi)(m) = \chi_{m_0}(m) = \exp(\pi\sqrt{-1}m_0)$  and  $\tau_{(-k, -k)}(m) = 1$  imply that  $m_0$  is an even integer.



**Proposition 3.1.** ([21, Proposition 7.2]) *Let  $\pi$  and  $\tau$  be as in §2.1. Then we have the following:*

(i) *We have  $\dim \mathcal{I}_{\eta_2, \pi} = \dim \text{SW}(\pi, \eta_2, \tau) \leq 4$  and a function*

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) h_{SW}(a)$$

*is in the space  $\text{SW}(\pi, \eta_2, \tau)|_A$  if and only if  $h_{SW}(a) = h_{SW}(y)$  is a smooth solution of following system:*

$$(3.1) \quad \left\{ \partial_{y_1} \left( -\partial_{y_1} + \partial_{y_2} + \frac{1}{2} \right) + \frac{y_1}{y_1 - 1} \left( -\partial_{y_1} + \frac{1}{2} \partial_{y_2} \right) + \frac{m_0^2}{4} \frac{y_1}{(y_1 - 1)^2} \right\} h_{SW}(y) = 0,$$

$$(3.2) \quad \left\{ \left( \partial_{y_2} + \frac{k+\nu}{2} \right) \left( \partial_{y_2} + \frac{k-\nu}{2} \right) - y_1 y_2 \left( \partial_{y_1} + \frac{1}{2} \right) - y_2 \left( -\partial_{y_1} + \partial_{y_2} + \frac{1}{2} \right) \right\} h_{SW}(y) = 0,$$

*with  $\partial_{y_i} = y_i(\partial/\partial y_i)$ .*

(ii)  $\dim \text{SW}(\pi, \eta_2, \tau)^{\text{rap}} \leq 1$ .

### 3.2 Explicit formulas of secondary Siegel-Whittaker functions

We consider the power series solution of the system in Proposition 3.1 around  $(y_1, y_2) = (0, 0)$ . In the notation in [16], this is the solution at  $Q_\infty$ .

**Proposition 3.2.** *For  $\nu \notin \mathbf{Z}$ , set  $h_{SW}^i(\nu; a) = h_{SW}^i(\nu; y)$  by*

$$h_{SW}^1(\nu; y) = (1 - y_1)^{|m_0|/2} \sum_{m, n \geq 0} c_{m, n}^1 \Gamma \left[ \begin{matrix} m - n + \frac{k+|m_0|-\nu}{2}, & m + \frac{|m_0|+1}{2} \\ -n + \frac{k+|m_0|-\nu}{2}, & \frac{|m_0|+1}{2} \end{matrix} \right] y_1^m y_2^{n+(-k+\nu)/2},$$

$$h_{SW}^2(\nu; y) = (1 - y_1)^{|m_0|/2} \sum_{m, n \geq 0} c_{m, n}^2 \Gamma \left[ \begin{matrix} m + n + \frac{-k+|m_0|+\nu}{2} + 1, & m + \frac{|m_0|+1}{2} & \frac{k+|m_0|-\nu}{2} \\ \frac{-k+|m_0|+\nu}{2} + 1, & \frac{|m_0|+1}{2} & n + \frac{k+|m_0|-\nu}{2} \end{matrix} \right] \\ \cdot y_1^{m+n+(-k+\nu+1)/2} y_2^{n+(-k+\nu)/2}.$$

Here  $c_{m, n}^1$  and  $c_{m, n}^2$  are the coefficients defined in Proposition 2.2. Then the power series  $h_{SW}^i(\nu; y)$  converges  $|y_1| < 1$  and  $y_2 \in \mathbf{C}$  and the set  $\{h_{SW}^i(\varepsilon\nu; y) \mid i = 1, 2, \varepsilon \in \{\pm 1\}\}$  forms a basis of the space of solutions of the system in Proposition 3.1.

### 3.3 Explicit formulas of good Siegel-Whittaker functions

The integral representation of the unique element in  $\text{SW}(\pi, \eta_2, \tau)^{\text{rap}}|_A$  is given by Miyazaki ([21, Theorem 7.5]). For our purpose, however, we need another integral expression for this function. Inspired by the work of Debiard and Gaveau ([1], [2]), we obtain the following Euler type integral. See also Iida ([15]) and Gon ([6]).

**Proposition 3.3.** *Define*

$$g_{SW}(a) = g_{SW}(y) := (1 - y_1)^{|m_0|/2} y_2^{|m_0|/2} \int_0^1 t^{(|m_0|-1)/2} (1-t)^{(|m_0|-1)/2} F\left(\frac{y_2}{2} \{1-t(1-y_1)\}\right) dt,$$

with

$$F(z) = e^z (2z)^{(-k-|m_0|-1)/2} W_{(k-|m_0|-1)/2, \nu/2}(2z).$$

Then the function

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) g_{SW}(a)$$

gives a non-zero element in  $SW(\pi, \eta_2, \tau)^{\text{rap}}|_A$  which is unique up to constant multiple.

*Proof.* See [6, 8.4]. □

### 3.4 Expansion theorem for Siegel-Whittaker functions

**Theorem 3.4.** For  $\nu \notin \mathbf{Z}$ , let  $h_{SW}^i(\nu; a)$  and  $g_{SW}(a)$  be the function defined in Proposition 3.2 and 3.3, respectively. Then

$$\begin{aligned} g_{SW}(a) = c_{SW} \sum_{\varepsilon \in \{\pm 1\}} & \left( \Gamma \left[ -\varepsilon\nu, \frac{-k + \varepsilon\nu + 1}{2} \right] h_{SW}^1(\varepsilon\nu; a) \right. \\ & \left. + \Gamma \left[ -\varepsilon\nu, \frac{k - \varepsilon\nu - 1}{2}, \frac{-k + |m_0| + \varepsilon\nu}{2} + 1 \right] h_{SW}^2(\varepsilon\nu; a) \right) \end{aligned}$$

with

$$c_{SW} = \Gamma \left[ \frac{|m_0| + 1}{2}, \frac{-k + |m_0| - \nu}{2} + 1, \frac{-k + |m_0| + \nu}{2} + 1 \right].$$

## 4 Confluences

### 4.1 Confluence of the differential equations

**Theorem 4.1.** If we substitute

$$(4.1) \quad h_1 = t^2 c_3, \quad h_2 = c_3, \quad m_0 = \frac{2\pi c_0}{t}$$

in the system in Proposition 3.1 and take the limit  $t \rightarrow 0$ , then we obtain the system in Proposition 2.1.

### 4.2 Confluence of the secondary spherical functions

**Theorem 4.2.** For  $\nu \notin \mathbf{Z}$ , define the functions  $h_{SW}^i(\nu, t; a)$  ( $i = 1, 2$ ) by substituting (4.1) in  $h_{SW}^i(\nu; a)$ . Then

$$\begin{aligned} \lim_{t \rightarrow 0} h_{SW}^1(\nu, t; a) &= h_W^1(\nu; a), \\ \lim_{t \rightarrow 0} t^{k-\nu-1} h_{SW}^2(\nu, t; a) &= h_W^2(\nu; a). \end{aligned}$$

### 4.3 Confluence of the spherical functions

**Theorem 4.3.** *Define the function  $g_{SW}(t; a)$  by substituting (4.1) in  $g_{SW}(a)$ . Then*

$$\lim_{t \rightarrow 0} \frac{g_{SW}(t; a)}{c_{SW}} = \frac{g_W(a)}{c_W}.$$

## 5 Deformation from $(R_2, \eta_2)$ to $(R_1, \eta_1)$ and the confluence

In this section we explain the main results in the previous section from the points of view of deformations and contractions of Lie groups (cf. [3]). This is to supply a heuristic background for the computations in the previous sections.

### 5.1 From $SO(2)$ to $N_0$

We first consider the deformation of two subgroups of  $SL(2, \mathbf{R})$ :

$$SO(2) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbf{R} \right\},$$

$$N_0 = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbf{R} \right\}.$$

Under the usual action of  $SL(2, \mathbf{R})$  to the upper half plane  $\mathfrak{h} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$ ,  $SO(2)$  is the stabilizer subgroup of  $\sqrt{-1}$  and  $N_0$  fixes  $\sqrt{-1}\infty$ . Set  $z_t = \sqrt{-1}/t$  for  $t > 0$ . Then  $\lim_{t \rightarrow \infty} z_t = \sqrt{-1}\infty$  and the stabilizer subgroup  $\text{Stab}_{SL(2, \mathbf{R})}(z_t)$  of  $z_t$  in  $SL(2, \mathbf{R})$  is

$$\begin{aligned} \text{Stab}_{SL(2, \mathbf{R})}(z_t) &= \left\{ r_\theta(t) = \begin{pmatrix} \cos \theta & \sin \theta/t \\ -t \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbf{R} \right\} \\ &= \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix} SO(2) \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}. \end{aligned}$$

For our purpose, we have to move  $\theta = \theta(t)$  such as  $\sin \theta(t)/t \rightarrow c$  as  $t \rightarrow 0$ . Let  $\theta(t) = ct$ . Then

$$(5.1) \quad r_{\theta(t)}(t) = \begin{pmatrix} \cos \theta(t) & \sin \theta(t)/t \\ -t \sin \theta(t) & \cos \theta(t) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

as desired.

### 5.2 From $R_2$ to $R_1$

Let  $\xi_t$  ( $t \neq 0$ ) be the definite character of  $N_S$  associated with  $H_{\xi_t} = \begin{pmatrix} ht^2 & 0 \\ 0 & h \end{pmatrix}$  ( $h > 0$ ):

$$\xi_t(n(0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1}(ht^2 n_1 + hn_3)).$$

Then the stabilizer subgroup  $SO(\xi_t)$  is identified with  $\{r_\theta(t) \mid t > 0\}$ . Therefore if we take  $\theta = \theta(t) = tn_0$  as in the previous subsection, (5.1) implies

$$(5.2) \quad \lim_{t \rightarrow 0} SO(\xi_t) = \{n(n_0, 0, 0, 0) \mid n_0 \in \mathbf{R}\}$$

in  $L_S$  makes up  $R_1 = N$  together with  $N_S$ , as we expected.

### 5.3 From $\eta_2$ to $\eta_1$

Define the character  $\chi_{m_0(t)}$  of  $SO(\xi_t)$  by

$$\chi_{m_0(t)}(r_\theta(t)) = \exp \sqrt{-1}(m_0(t)\theta).$$

and put  $\eta_{2,t} = \chi_{m_0(t)} \boxtimes \xi_t$ .

$$\eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \exp 2\pi\sqrt{-1}(ht^2n_1 + hn_3) \cdot \exp \sqrt{-1}(m_0(t)\theta(t)).$$

Since  $\theta(t) = n_0t$ , we should take

$$m_0(t) = \frac{2\pi c_0}{t}.$$

Then the right hand side goes to  $\exp 2\pi\sqrt{-1}(c_3n_3 + c_0n_0)$  (after the replacement  $h = c_3$ ), and thus combined with (5.2), we obtain

$$\lim_{t \rightarrow 0} \eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \eta_1(n(n_0, n_1, n_2, n_3)).$$

**Remark 2.** Our result should be regarded as the investigation of the intertwining spaces:

$$\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(H_{\pi, K}, C_{\eta_{2,t}}^\infty(R_t \backslash G))$$

with

$$R_t = SO(\xi_t) \ltimes N_s \quad (t > 0).$$

## 6 Further comments

We only treat the even  $P_J$ -principal series, however, we also have the same results for the odd case, that is,  $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = -(-1)^k$ .

In the case of the principal series (induced from minimal parabolic subgroup of  $G$ ), the holonomic systems of rank 8 for the radial part of Whittaker functions (resp. Siegel-Whittaker functions) are obtained in [22] (resp. [21], [16]) and we can prove the same assertion as Theorem 4.1. However, explicit formulas for secondary spherical functions are known only for Whittaker functions ([17]), we can not say any more.

The other kinds of spherical functions on  $Sp(2, \mathbf{R})$  are studied by Moriyama ([24]) and by Hirano ([12], [13], [14]). The spherical subgroup of [24] is  $SL(2, \mathbf{C})$  and of [12], [13] and [14] is  $SL(2, \mathbf{R}) \ltimes H_3$ , with  $H_3$  the 3-dimensional Heisenberg group. We hope that similar results hold between the two spherical functions.

We finally remark on the case of the special unitary group  $SU(2, 2)$ , which has the same restricted root system as  $Sp(2, \mathbf{R})$ . (Siegel-) Whittaker functions on  $SU(2, 2)$  are studied by Hayata and Oda ([10], [8], [9]) and by Gon ([6]). Since their differential equations are compatible to those of  $Sp(2, \mathbf{R})$ , analogous argument seems to be possible.

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